This is an expository note describing the isomorphism classes of \((\text{algebraic})\) line bundles on projective spaces in an elementary way. Our intention is to introduce the isomorphism classes of line bundles on a projective space in three different ways

- By projective geometry,
- By local coordinates,
- As multiplicative group quotients.

0.1. Notations and Conventions. During this note, we will fix a base field \(k\) (e.g. the complex numbers \(\mathbb{C}\)). The multiplicative group of the field \(k\) will be denoted by \(\mathbb{G}_m\). All the vector spaces considered will be finite dimensional vector spaces over the field \(k\).

For a given vector space \(V\) of dimension \(\geq 1\), we will denote the dual vector space, the projective space parameterizing \(1\)-dimensional subspaces in \(V\) and the projective space parameterizing subspaces of codimension 1 in order by \(V^\vee\), \(\mathbb{P}(V)\) and \(\mathbb{P}(V^\vee) = \mathbb{P}(V)\). By the projectivization of the vector space \(V\), we will mean the projective space \(\mathbb{P}(V)\). The projective space \(\mathbb{P}(V^\vee)\) is called the dual projective space. Notice that the dimensions of the projective space \(\mathbb{P}(V)\) and \(\mathbb{P}(V^\vee)\) are one less than that of the vector space \(V\). More generally, one can fix an integer \(l \leq \dim(V)\) and consider the variety parameterizing \(l\)-planes in the vector space \(V\). The resulting variety is called the Grassmannian of \(l\)-planes in \(V\) and is denoted by \(\text{Grass}(l, V)\). If \(L\) is a line in the vector space \(V\), we will denote the corresponding point in the projective space by \([L \subset V]\), or \([L]\) for short. More generally, if \(S\) is an \(l\)-dimensional plane in \(V\), the corresponding point in the Grassmannian \(\text{Grass}(l, V)\) will be denoted by \([S \subset V]\), or \([S]\) for short. Notice that by definition, the equalities \(\text{Grass}(1, V) = \mathbb{P}(V)\) and \(\text{Grass}(\dim(V) - 1, V) = \mathbb{P}(V^\vee)\) hold.

The isomorphism classes of line bundles on a smooth variety \(X\) is denoted by \(\text{Pic}(X)\). Under the tensor product, the set \(\text{Pic}(X)\) becomes an Abelian group, where the trivial element of the group is the trivial bundle, \(\mathcal{O}_X\) and the inverse of a line bundle is determined by the equation \(L \otimes L^{-1} = \mathcal{O}_X\). Therefore, for a given line bundle \(L\), the dual bundle is called the inverse of \(L\) or \(L^{-1}\), and it is denoted by \(L^{-1}\). For arbitrary integer \(n\), we define the \(n\)th multiple of the line bundle \(L\) by the formula:

\[
L^\otimes n := \begin{cases} 
L \otimes L \otimes \cdots \otimes L \quad &\text{if } n > 0; \\
\mathcal{O}_X \quad &\text{if } n = 0; \\
L^{-1} \otimes L^{-1} \otimes \cdots \otimes L^{-1} \quad &\text{if } n < 0.
\end{cases}
\]

The projective space comes equipped with two line bundles, called the universal line bundle and the hyperplane bundle, denoted by \(\mathcal{O}_{\mathbb{P}(V)}(-1)\) and \(\mathcal{O}_{\mathbb{P}(V)}(1)\) respectively. The significance of these bundles will be explained later. For the moment, we only need to say these two line bundles are duals of each other. Moreover, for a given integer \(d\), we define the line bundle \(\mathcal{O}_{\mathbb{P}(V)}(d)\) to be \(\mathcal{O}_{\mathbb{P}(V)}(1)^\otimes d\). The integer \(d\) is called the degree of the line bundle \(\mathcal{O}_{\mathbb{P}(V)}(d)\).
0.2. Preliminaries. In this section we want to touch few elementary, yet important details on vector spaces, affine spaces and projective spaces, and their bundle theoretic analogs. In addition, the definitions of a principal $G$-bundle and the associated vector bundles will be provided.

One feature of vector spaces is that every vector space comes with a distinguished point, namely the origin, $0_V \in V$. It is also important not to forget that no vector space has a canonical basis, and every basis is, indeed, a choice of vectors in the vector space $V$. One main concept in linear algebra is the linear combination. The endomorphisms of a vector space are defined to be those preserving the linear structure, i.e. preserving linear combinations, called the linear transformations. Those which are also invertible will be called as the linear automorphisms, meaning that automorphism of the linear structure. The resulting group is called the general linear group and for a given vector space it will be denoted by $GL(V)$.

For a given vector space $V$, the affine space $A(V)$ is defined to be a space with two properties:

- Its set of points is the same as $V$,
- On this set, $V$ acts by translations.

We can call the above conditions as an affine structure on the related space, and call such a space as an affine space associated to $V$. Technically precise definition is that an affine space associated to $V$ is a principal homogeneous space for the additive group $(V, +)$. Note that under the translations, $T_v$, sending a point $x$ to $x + v$, there is no fixed point, hence there is no preferred points as in the case of the vector space $V$. Similar to linear algebra, one can define the notion affine combination of points $x_1, \ldots, x_m$ as $a_1 x_1 + \cdots + a_m x_m$ where $a_1 + a_2 + \cdots + a_m = 1$. An affine transformation is the one which preserves the affine combinations. If in addition, it is invertible, such a transformation is called an affine transformation, or equivalently, an automorphism of the affine structure. The resulting automorphism group will denoted by $AGL(V)$. The notation we have for the affine automorphism group is not standard. Every element, $T_v$, of the group $AGL(V)$ can be written as $T(\cdot) = S(\cdot) + v$ where $S \in GL(V)$ and $v \in V$. The group $AGL(V)$ is a semi-direct product of the group $GL(V)$ by the additive group $(V, +)$ and hence fits in the sequence:

$$0 \to V \to AGL(V) \to GL(V) \to 1$$

We again remind the following subtle point: Every vector space comes equipped with a distinguished point (the origin $0_V$), but not an affine space.

Third one in the list is the projective space associated to $V$. Although, they are related to linear and affine structures, projective structures are quite different from those two. There are several ways to think about the projective space $\mathbb{P}(V)$.

By definition, it is the variety parameterizing 1-dimensional linear subspaces in $V$. Or, equivalently, we can view it as the quotient, $[V \setminus \{0\}] / \mathbb{G}_m$. The automorphisms in the general linear group $GL(V)$, which preserve a fixed line $L \subset V$ is those of the form $\lambda d_v$ for $\lambda \in \mathbb{G}_m$. Actually, such transformations form the center of the group $GL(V)$. The resulting quotient is called the projective linear group and denoted by $PGL(V)$. It is the automorphism group of the projective space $\mathbb{P}(V)$. Then, we have the sequence

$$1 \to Z \to GL(V) \to PGL(V) \to 1$$

On a given projection $p : T \to B$, one can define a linear (resp. affine, resp. projective) structure if there exists a covering of the base space $B$ by open sets $U_i$, so that on such a typical open set $U_i \subset B$, one can find a local trivialization $h_i : p^{-1}(U) \to k^n \times U_i$ (resp. $A(k^n)$, resp. $P(k^n)$) so that resulting transition function $g_{ij} = h_i \circ h_j^{-1}$ is in the group $GL(V)$ (resp. $AGL(V)$, resp. $PGL(V)$).

Such a bundle $p : T \to B$ will be called a vector bundle (resp. affine bundle, resp. projective bundle). The definition of affine and projective bundles in the literature are a bit broader than
our description given here. For example, our definition for a projective bundle will be called a projectivized vector bundle. However, for the purposes of this paper, our terminology should be fine.

The difference between a vector space and the associated affine space due to having a distinguished point, also shows up in the bundle theoretic frame. A vector bundle has a section from the beginning which consists of zero elements in the fibers. It is called the zero section and will be denoted by \( Z \), throughout this paper. An affine bundle may have global sections, but none of them is distinguished like the zero section of a vector bundle. In the case of affine space, fixing a point in the affine space, identifies the affine space and the vector space on which it is defined as sets. Via the set-theoretic identification, one can induce a linear structure on the affine space isomorphic to that of the vector space defining it. Notice, all has to do is to name a point as the zero element. Similarly, we will see that if an affine bundle has global sections, in some cases one is able to induce a linear structure on it, by simply declaring one of these global sections as the zero section. Nevertheless, our example is a very specific one, and it does not represent the general nature of affine bundles.

Given an arbitrary Lie group \( G \), one can consider the fiber bundles whose structure group is \( G \). In addition to the group \( G \), fix a variety \( F \) on which \( G \) acts on the right. In the notation of the previous definition, the projection \( p : T \to B \) is a fiber bundle with fiber \( F \) and the structure group \( G \), if the trivializations \( h_i \) identify the spaces \( p^{-1}(U_i) \to F \times U_i \) so that the transition functions \( g_{ij} \) are (algebraic) mappings from the overlaps \( U_i \cap U_j \) to the group \( G \). If the fiber \( F \) is the group itself, the resulting bundle is called a principal \( G \)-bundle, and usually denoted by \( E_G \).

Given a representation \( \rho : G \to GL(V) \), i.e. the group \( G \) acts on the vector space by linear automorphisms on the left, one can construct the associated vector bundle by the recipe: Construct the Cartesian product \( E_G \times V \), take the quotient by identifying the pairs \( (e, v) \) and \( (eg, \rho(g)^{-1} \cdot v) \) for all \( g \in G \). The resulting bundle will be denoted by \( E_G \times_V V \), or \( E_G(\rho) \), for short. We will come across only one principal bundle, a principal \( G_m \)-bundle obtained from the projection \( p : V \setminus 0 \to \mathbb{P}(V) \) and its associated line bundle for the given character \( \chi : G_m \to G_m \), which will be denoted by \( O_{\mathbb{P}(V)}(\chi) \).

0.3. Outline. In Sect. 1, we recollect some of the natural constructions in linear algebra along with their counterparts in projective geometry. The ultimate principle is that every natural construction in linear algebra induces a construction in projective geometry. However, not all constructions in projective geometry following the principle above. Subsection 1.14 is devoted to one such construction related to the natural projection from the vector space \( V \) to its projectivization \( \mathbb{P}(V) \). Although, it is not given by the above principle, this map is closely related to the quotient construction of the Section 1. As we will see later, the natural projection \( V \to \mathbb{P}(V) \) is not a map in the strict sense and the effort to fix this problem will bring us to another fundamental operation in projective algebraic geometry, namely the blowup. The universal line bundle, \( O_{\mathbb{P}(V)}(-1) \) on the projective space \( \mathbb{P}(V) \) will be obtained via the blowup.

For a given line bundle \( L \) on a smooth variety \( X \), we will denote the set of global sections by \( H^0(X, L) \). Every line bundle has a trivial section, namely the zero section sending the points of \( X \) to the zero element in the fiber above. For a vector bundle, the set of global sections is a vector space, since one can add two sections and multiply by scalars due to linear structure of the bundle.

1. Linear Algebra vs. Projective Geometry

In mathematics, one tries to resort to natural constructions as much as possible. One particular reason in the case of algebraic geometry is the fact that it is difficult to introduce local coordinates on algebraic varieties, even on smooth ones. The other is a very general principle of mathematics saying that “The natural is Good. The functorial is better”. We know that the natural constructions
on vector spaces (e.g. direct sum, wedge powers etc.) induce constructions on vector bundles, since they do not involve any kind of extra choice. Being independent of choice is the implicit key idea in the above expression. In the case of a vector space, the choice one can make is the choice of a basis. To emphasize the importance of naturally, we point out how it shows up in linear algebra with two simple examples.

While one can talk about the naturality of constructions resulting in objects, one can also talk about the naturality of constructions resulting in maps between two objects. Our examples will be two linear maps, indeed isomorphisms. We want to observe how the state of naturality determines when passing from a pair of vector spaces to a pair of vector bundles. Notice that the first one is natural, the second is not:

\[ V \to V^{\vee \vee} \]
\[ V \to V^{\vee} \]

The map in line (1) is defined by sending a vector \( v \in V \) to the evaluation map \( \text{ev}_v \in V^{\vee \vee} \) where the evaluation map \( \text{ev}_v \) acts on a linear functional \( f \in V^{\vee} \) by sending it to \( f(v) \in k \), i.e. \( \text{ev}_v(f) := f(v) \).

The resulting map \( V \to V^{\vee \vee} \) will be denoted by \( \text{ev} \). We say that the map \( \text{ev} \) is a natural linear map (isomorphism) and the vector space \( V \) and its double dual \( V^{\vee \vee} \) are naturally isomorphic, because the map \( \text{ev} \) is defined without the choice any particular basis.

Again, it is part of linear algebra proper that all vector spaces of a given dimension \( n \) are isomorphic. This is the way we construct the map in line (2), by choosing a basis \( v_1, v_2, \ldots, v_n \) in \( V \) and a dual basis \( v_1^{\vee}, v_2^{\vee}, \ldots, v_n^{\vee} \) in \( V^{\vee} \) so that \( v_i^{\vee}(v_j) = \delta_{ij} \) where \( \delta_{ij} \) is the Kronecker delta function. The map sends the basis vector \( v_i \) to its dual \( v_i^{\vee} \) for \( i = 1, \ldots, n \). By its definition, the isomorphism \( V \to V^{\vee} \) is not natural. Indeed, it is a nice exercise to show that there cannot be any natural isomorphism between the vector space \( V \) and its dual \( V^{\vee} \).

The outcome of the naturality of the map \( \text{ev} \) is that it extends to vector bundles. Given a vector bundle \( V \), there is an evaluation map (denoted by \( \text{ev} \) again) from \( V \) to its double dual \( V^{\vee \vee} \). Moreover, the bundle map \( \text{ev} : V \to V^{\vee \vee} \) is an isomorphism, and we say similarly as before that the bundles \( V \) and \( V^{\vee \vee} \) are naturally isomorphic. However, we see that a vector bundle \( V \) is rarely isomorphic to its dual \( V^{\vee} \). For example, if this were true, the global sections sections of the bundle should be the same as the other. However, it is easy to construct examples where the bundle \( V \) has no global sections and on the contrary, the dual \( V^{\vee} \) will have many. We will construct such examples in the sequel, e.g. the line bundles \( \mathcal{O}_{\mathbb{P}(V)}(-1) \) and \( \mathcal{O}_{\mathbb{P}(V)}(1) \) on a given vector space \( \mathbb{P}(V) \) are examples of such a pair.

After emphasizing the importance of naturality in linear algebra, let’s see how these ideas induce constructions in projective geometry.

1.1. Linear Maps. The natural step after projectivizing vector spaces is to projectivize the the linear maps between vector spaces. However, every linear map between two vector spaces factors through the image of itself and becomes a composition of a surjection followed by an injection. Motivated by this, we will concentrate on the extremes, injective and surjective linear maps separately. The general case will then follow from these two extremes described as above. In the sequel, by making the necessary identifications, we will concentrate on the natural injections (=subspaces) and the natural surjections (=quotients by subspaces).

1.2. Subspaces. Suppose \( W \) is a nontrivial subspace of \( V \). Obviously, a line in \( W \) is also a line in \( V \). So we have an inclusion of projective spaces \( \mathbb{P}(W) \hookrightarrow \mathbb{P}(V) \) by sending the class of the line \( L \) in \( W \) to the class of the line \( L \) in \( V \). The subspace \( \mathbb{P}(W) \) is called a linear subspace of the projective space \( \mathbb{P}(V) \). Note that all linear subspaces, i.e. the subspaces cut out by a set of homogeneous linear equations, are obtained this way.
1.3. **Quotients.** Suppose the quotient \( V/W \) of the vector space \( V \) by the subspace \( W \) is of dimension \( \geq 1 \). Consider the projectivization of the quotient map \( V \to V/W \). Given a line \( L \) in \( V \), \([L]\) will be sent to the subspace generated by \( W \) and \( L \) modulo \( W \), \([L+W]/W\]. However, there is one problem with this association. If the line \( L \) happens to lie in the subspace \( W \), then \( L+W = W \) hence \( (W+L)/W \) is the trivial subspace \( \langle 0_{V/W} \rangle \) rather than being a line in the quotient space \( V/W \). The locus along which the projectivized quotient map is not well-defined is exactly the linear subspace \( \mathbb{P}(W) \). The map \( \mathbb{P}(V) \\setminus \mathbb{P}(W) \to \mathbb{P}(V/W) \) is called the projection with the center \( \mathbb{P}(W) \). It will be denoted by \( \pi_{\mathbb{P}(W)} \) or \( \pi_W \).

The projectivized quotient map from \( \mathbb{P}(V) \) to \( \mathbb{P}(V/W) \) is our first example of a rational map.

**Definition 1.4.** Let \( X \) and \( Y \) be two varieties. Suppose \( \varphi \) is a morphism from a dense subset of \( X \) to the variety \( Y \), which possibly does not extend to all of \( X \). Let \( U \) be the set of points for which \( \varphi \) is well-defined. Then the map \( \phi : U \to Y \) is called a rational map from \( X \) to \( Y \) and denoted by \( \phi : X \dashrightarrow Y \). The locus \( X \setminus U \) is called the indeterminacy locus of the rational map \( \varphi \).

The indeterminacy locus of the rational map \( \pi_{\mathbb{P}(W)} : \mathbb{P}(V) \to \mathbb{P}(V/W) \) is exactly the subspace \( \mathbb{P}(W) \). Later, we will come across another example of a rational map.

1.5. **Tensor Powers.** The general linear group \( GL(V) \) acts on the vector space \( V \) on the left. Consequently, it acts on the \( d \)th tensor power \( V^\otimes d \) on the left also. In addition to the general linear group \( GL(V) \), the symmetric group on \( d \) letters, \( S_d \), acts on \( V^\otimes d \), say on the right, by permuting the components of a decomposable tensor. In other words, for a given permutation, \( \sigma \in S_d \), define \( (v_1 \otimes v_2 \otimes \cdots \otimes v_d) \cdot \sigma := v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(d)} \) and extend it linearly to the indecomposable tensors.

The important fact concerning the representation theory of the groups \( GL(V) \) and \( S_d \) on \( V^\otimes d \) is that they have the same irreducible subspaces. The two most important of these subspaces are the \( d \)th symmetric power, \( \text{Sym}^d V \), and the \( d \)th wedge product, \( \wedge^d V \), which are defined as follows:

\[
\begin{align*}
\text{Sym}^d V &:= \{ w \in V^\otimes d \mid w \cdot \sigma = w \quad \text{for all } \sigma \in S_d \} \\
\wedge^d V &:= \{ w \in V^\otimes d \mid w \cdot \sigma = \text{sgn}(\sigma)w \quad \text{for all } \sigma \in S_d \}
\end{align*}
\]

where \( \text{sgn}(\sigma) \) denotes the sign of the permutation \( \sigma \in S_d \).

For further details of the representation theory of the tensor powers, Young symmetrizers, Schur Functors and Weyl’s construction, see the book [FH].

1.6. **Symmetric Powers.** For us, the significance of the \( d \)th symmetric power \( \text{Sym}^d V \) comes from the following: First, notice that the \( d \)th tensor power of the vector \( v \in V \), the tensor \( v^\otimes d \) is preserved under the action of the symmetric group \( S_d \) since all of its components are the same. Thus, \( v^\otimes d \in \text{Sym}^d V \). As a result, we see that the map \( V \to V^\otimes d \) sending the vector \( v \in V \) to its \( d \)th tensor power \( v^\otimes d \in V^\otimes d \) factors through \( \text{Sym}^d V \). Consequently, the projectivized map \( \mathbb{P}(V) \to \mathbb{P}(V^\otimes d) \) factors through the projective space \( \mathbb{P}(\text{Sym}^d V) \) and equivalently, the diagram commutes

\[
\begin{array}{ccc}
\mathbb{P}(V) & \rightarrow & \mathbb{P}(V^\otimes d) \\
\downarrow & & \downarrow \\
\mathbb{P}(\text{Sym}^d V) & \rightarrow & \mathbb{P}(\text{Sym}^d V)
\end{array}
\]

The map \( \mathbb{P}(V) \to \mathbb{P}(\text{Sym}^d V) \) goes under the name \( d \)th Veronese embedding and will be denoted by \( \vartheta_d \). This map and its highly nonlinear nature will prove very useful in Sect. 5.

In order to enlighten some of the properties of the Veronese embedding, \( \vartheta_d : \mathbb{P}(V) \to \mathbb{P}(\text{Sym}^d V) \), here we include some exercises on the degree of subvarieties in a given projective space.

**Definition 1.7.** The degree of a projective variety \( X \) embedded in a projective space \( \mathbb{P}^n \) is the number of points of the intersection of the variety \( X \) with the generic plane of the complementary dimension.
Exercise 1.8. Show that for a curve $X$ in $\mathbb{P}^2$, the degree of $X$ is equal to the degree of the homogeneous polynomial defining the curve $X$. Prove this equality for arbitrary hypersurface in the projective space $\mathbb{P}^n$. Conclude that if $X$ is a hypersurface of degree 1, it is a hyperplane in $\mathbb{P}^n$.

Exercise 1.9. Assume that the projective space $\mathbb{P}(V)$ is embedded in another projective space $\mathbb{P}(W)$ by $\varphi$. Show that the degree of the image of the embedding $\varphi$ is equal to the degree of the hypersurface $\varphi^{-1}(H)$ where $H$ is a hyperplane $\mathbb{P}(W)$.

Exercise 1.10. Try to calculate the degree of the image of the map $\mathbb{P}(V) \hookrightarrow \mathbb{P}({\text{Sym}}^d V)$ with coordinate free arguments. (Hint: It may be helpful to do with coordinates first. Though, we strongly suggest that it is one’s benefit to think in a coordinate-free way.)

1.11. Wedge Powers. Previously, we have seen that taking $d$th wedge power is not a good idea: The result is 0 if $d \leq 2$. However, if we fix a $d$-dimensional subspace $S$ in $V$, we can take the $d$th wedge power of the inclusion $S \hookrightarrow V$ and get the inclusion $\wedge^d S \hookrightarrow \wedge^d V$. Notice that, $\dim \wedge^d S = 1$, i.e. $\wedge^d S$ is a line in $\wedge^d V$. In other words, we obtain a map from the Grassmannian $\text{Grass}(d, V)$ into $\mathbb{P}(\wedge^d V)$ sending $[S \subset V]$ to $[\wedge^d S \subset \wedge^d V]$. The resulting map is called the Plücker embedding. It is used to prove that the Grassmannian and all the flag varieties are projective varieties.

1.12. Tensor Product. Although we have touched on tensor powers of a given vector space $V$, we have not look at the tensor operation in general. The projective space $\mathbb{P}(V \otimes W)$ shows up in the proof of the statement that the product of two projective varieties is again projective. A step in the proof is to show that the above statement is true for the product of two projective spaces $\mathbb{P}(V)$ and $\mathbb{P}(W)$. The product $\mathbb{P}(V) \times \mathbb{P}(W)$ embeds in the projective space $\mathbb{P}(V \otimes W)$ by sending a pair of lines $L$ and $M$ into their tensor product $L \otimes M$. The above map is coined the name , the Segre embedding.

Exercise 1.13. One naive idea is to try to send the product $\mathbb{P}(V) \times \mathbb{P}(W)$ into the projectivization of the direct sum, $\mathbb{P}(V \oplus W)$. In general, this is only a rational map. That is, it is not defined everywhere on the domain $\mathbb{P}(V) \times \mathbb{P}(W)$. Describe how to resolve the indeterminacies of the rational map $\mathbb{P}(V) \times \mathbb{P}(W) \dashrightarrow \mathbb{P}(V \oplus W)$. What is the relation between $\mathbb{P}(V) \times \mathbb{P}(W) \dashrightarrow \mathbb{P}(V \oplus W)$ and the Segre embedding, $\mathbb{P}(V) \times \mathbb{P}(W) \hookrightarrow \mathbb{P}(V \otimes W)$ ?

1.14. The Missing Link: $V \dashrightarrow \mathbb{P}(V)$. So far, we have considered how a linear algebra construction induced a projective geometry construction. What about the relation between the spaces $V$ and $\mathbb{P}(V)$ ?

One sees that there is an obvious map, sending a vector $v \in V$ to the line spanned by $v$, $[(v) \subset V]$. We see that this is another example of a rational map, well-defined on the subvariety $V-\{0\}$. Notice that the fibers of this map are lines minus the origin. Hence, they are copies of the multiplicative group $\mathbb{G}_m$. Later, we will show that the projection $p : V-\{0\} \to \mathbb{P}(V)$ defines a principal $\mathbb{G}_m$-bundle over the projective space $\mathbb{P}(V)$ which will be denoted by $E_{\mathbb{G}_m}$. Since our concern is to locate the line bundles on projective spaces rather than the principal $\mathbb{G}_m$-bundles, we will not pursue this direction in depth. However, we will relate the above principal bundle to the line bundles on the projective space $\mathbb{P}(V)$.

Already the map $p : V-\{0\} \to \mathbb{P}(V)$ is very close to a line bundle, the only problem is that every fiber is missing a zero element. The way to fix this trouble is a very general to tool to resolve indeterminacies of rational maps called the blow up. The name is suggested by the fact, the map $p$ at the origin is many-valued. Since all nonzero vector on a line $L$ are sent to the point $[L \subset V]$ in $\mathbb{P}(V)$, if the map were defined at the origin, the continuity requires that the origin should also be sent to the point $[L \subset V]$. However, this is true for every line $L$ in $V$. So we see that if the map were defined at the origin, it has to sent 0 onto the projective plane $\mathbb{P}(V)$ by continuity. In other words, the map $p$ blows up at the origin. This is why the corresponding construction is called the
blow up. We will come over this trouble by another standard technique in algebraic geometry: The incidence varieties.

We consider the variety \( T \) defined in the product \( V \times \mathbb{P}(V) \) by the rule: The pair \( (v, [L]) \in T \) if and only if \( v \in L \). The variety \( T \) is an example of incidence varieties. The following diagram is nice way to put things together.

\[
\begin{array}{ccc}
T & \xrightarrow{\beta} & V \\
\downarrow{\pi} & & \downarrow{\pi} \\
\mathbb{P}(V) & & \mathbb{P}(V)
\end{array}
\]

The left hand side of the diagram is called the blowup of \( V \) at the origin. The right hand side is a line bundle. It is called the universal line bundle on the projective space \( \mathbb{P}(V) \) and denoted by \( O_{\mathbb{P}(V)}(-1) \). It is also called the tautological line bundle or the universal subbundle. The latter name comes from the fact that one can similarly define an incidence variety \( T \) for the Grassmannian \( \text{Grass}(d, V) \) of \( d \)-planes in \( V \). Then fiber of \( T \) above the point \( [S \subset V] \) is the subspace \( S \) itself. If one thinks about it for a while, it is not hard to see that \( T \) is universal with respect to the above property. We will denote the universal subbundle by \( S \), then \( T = \text{Tot}(S) \). We will denote The trivial bundle whose fibers are \( V \) will be denoted by \( V \otimes \mathcal{O} \), hence \( \text{Tot}(V \otimes \mathcal{O}) = V \times \mathbb{P}(V) \).

We have defined the universal subbundle \( S \) via defining its total space \( T \) in the total space of the trivial \( V \)-bundle, \( V \times \mathbb{P}(V) \). As a result, we will get an inclusion of bundle \( S \hookrightarrow V \otimes \mathcal{O} \). Naming the quotient by \( S \), we have arrived at the exact sequence

\[
0 \to S \to V \otimes \mathcal{O} \to \mathcal{O} \to 0
\]  

The above sequence is called the universal sequence.

In the case of the projective space \( \mathbb{P}(V) \), above diagram is

\[
0 \to O_{\mathbb{P}(V)}(-1) \to V \otimes \mathcal{O} \to \mathcal{O} \to 0
\]  

1.15. **Summary.** In this section, we have seen many constructions in projective geometry. Most of these induced by a linear algebra construction. Another one relating the vector space \( V \) to its projectivization \( \mathbb{P}(V) \) resulted in the universal line bundle \( O_{\mathbb{P}(V)}(-1) \). We want to summarize all of these results as follows:

<table>
<thead>
<tr>
<th>Linear Algebra</th>
<th>Projective Geometry</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vector space ( V )</td>
<td>Projective Space ( \mathbb{P}(V) )</td>
</tr>
<tr>
<td>Dual Vector Space ( V^\vee )</td>
<td>Dual Projective Space ( \mathbb{P}(V^\vee) )</td>
</tr>
<tr>
<td>Linear Subspace ( W \hookrightarrow V )</td>
<td>Proj. Linear Subspace ( \mathbb{P}(W) \hookrightarrow \mathbb{P}(V) )</td>
</tr>
<tr>
<td>Quotient Space ( V \hookrightarrow V/W )</td>
<td>Projection away from ( \mathbb{P}(W) ) ( \mathbb{P}(V) \setminus \mathbb{P}(W) \xrightarrow{\pi_{\mathbb{P}(W)}} \mathbb{P}(V/W) )</td>
</tr>
<tr>
<td>Tensor power ( V \otimes d ) ( v \in V \mapsto v \otimes d \in V \otimes d )</td>
<td>( [L \subset V] \mapsto [L \otimes d \subset V \otimes d] )</td>
</tr>
<tr>
<td>Symmetric Power ( \text{Sym}^d V ) ( v \in V \mapsto v \otimes d \in \text{Sym}^d V )</td>
<td>Veronese : ( \mathbb{P}(V) \setminus \mathbb{P}(\text{Sym}^d V) ) ( [L \subset V] \mapsto [\text{Sym}^d L \subset \text{Sym}^d V] )</td>
</tr>
<tr>
<td>Wedge Power ( \Lambda^d V )</td>
<td>Plucker : ( \text{Grass}(1,V) \setminus \mathbb{P}(\Lambda^1 V) ) ( [S \subset V] \mapsto [\Lambda^1 S \subset \Lambda^1 V] )</td>
</tr>
<tr>
<td>Tensor Product ( V \otimes W )</td>
<td>Segre : ( \mathbb{P}(V) \times \mathbb{P}(W) \setminus \mathbb{P}(V \otimes W) ) ( ([L], [M]) \mapsto [L \otimes M] )</td>
</tr>
<tr>
<td>( V \hookrightarrow \mathbb{P}(V) )</td>
<td>Universal Line Bundle ( O_{\mathbb{P}(V)}(-1) )</td>
</tr>
</tbody>
</table>

For this section, one of the best references is the book [DS].
2. The Conceptual Interlude

In this section, we address the question “what the line bundles, $O_{P(V)}(-1)$ and $O_{P(V)}(1)$ parameterize”.

As the first step, let’s clarify the same question for the bundles, $V \otimes \mathcal{O}$ and $V^\vee \otimes \mathcal{O}$. The former parameterizes the set of points of the vector space $V$, regardless of the choice of the base point in the projective space $\mathbb{P}(V)$. Similarly, the latter corresponds to the trivial family of linear functions (forms) on the vector space $V$. To sum up, looking at the fibers of these two bundles above a point $[L \subset V]$ in $\mathbb{P}(V)$ is enough:

$$(V \otimes \mathcal{O})|_{[L]} = V \quad \text{and} \quad (V^\vee \otimes \mathcal{O})|_{[L]} = V^\vee.$$

Unlike the two bundles above, the line bundles, $O_{P(V)}(-1)$ and $O_{P(V)}(1)$ depend on the point chosen in the projective space $\mathbb{P}(V)$. The universal line bundle, $O_{P(V)}(-1)$, parameterizes the set of points of the lines in $V$. The hyperplane bundle, $O_{P(V)}(1)$ parameterizes the set of linear functions on the lines in $V$.

For a given line $L$ in $V$, the fiber of the universal line bundle, $O_{P(V)}(-1)$, above the point $[L \subset V]$ is the set of points of the line $L$. In other words, the line bundle $O_{P(V)}(-1)|_{[L]} = L$. Similarly, the fiber of the hyperplane bundle, $O_{P(V)}(1)$, above the point $[L \subset V]$ is the space of linear functions on the line $L$, i.e., $L^\vee$. Equivalently, $O_{P(V)}(1)|_{[L]} = L^\vee$.

The inclusion of the universal line bundle in the trivial $V$-bundle, $O_{P(V)}(-1) \hookrightarrow V \otimes \mathcal{O}$, is to say “Forget that a vector $v$ is on the line $L$ and view it only as a vector in the space $V$”. For a fixed line $L$ in the vector space $V$, the map from the trivial $V^\vee$-bundle to the hyperplane bundle, $V^\vee \otimes \mathcal{O} \rightarrow O_{P(V)}(1)$, reads as “Restrict the linear functions on the space $V$ to the line $L$.”

In some way the universal sequence

$$0 \rightarrow O_{P(V)}(-1) \rightarrow V \otimes \mathcal{O} \rightarrow \mathcal{O} \rightarrow 0$$

and the dual

$$0 \rightarrow \mathcal{O}^\vee \rightarrow V^\vee \otimes \mathcal{O} \rightarrow O_{P(V)}(1) \rightarrow 0$$

are self-explanatory looking at the their fibers above the point $[L \subset V]$ in $\mathbb{P}(V)$:

$$0 \rightarrow L \rightarrow V \rightarrow V/L \rightarrow 0$$

$$0 \rightarrow (V/L)^\vee \rightarrow V^\vee \rightarrow L^\vee \rightarrow 0$$

3. The Universal Line Bundle: $O_{P(V)}(-1)$

The universal line bundle is defined via the diagram below. Now we briefly summarize the circle of ideas run in this section.

The left hand side of the diagram is the blow up of $V$ at the origin. Recall that, the map $p : V \setminus 0 \rightarrow \mathbb{P}(V)$ describes only a principal $\mathbb{G}_m$-bundle. However, like every vector bundle, a line bundle comes with a zero section. This is what is missing in the case of $V \setminus 0 \xrightarrow{p} \mathbb{P}(V)$ is the zero section. The zero section is added to the principal $\mathbb{G}_m$-bundle via blowing up the map $p$ along its indeterminacy locus, the origin. The exceptional divisor of the blowup then will be the zero divisor of the bundle $T \xrightarrow{\pi} \mathbb{P}(V)$.

$$\begin{array}{c}
T \\
\beta \downarrow \quad \pi \\
V \xrightarrow{p} \mathbb{P}(V)
\end{array}$$
In order to analyze a diagram like the one above, one must study the two projection maps separately, with the hope that some piece of information obtained about one of them, will help to understand the other one. This way, one aims to understand the relation between the varieties in the second row, e.g. the varieties $V$ and $\mathbb{P}(V)$ in the example above.

We will start analyzing this diagram from the right hand side. First, let’s identify the fibers of $\pi : T \to \mathbb{P}(V)$. Pick a point $[L \subseteq V]$ in $\mathbb{P}(V)$, the fiber $\pi^{-1}([L])$ is defined by $\{(v, [L])|v \in L\}$, i.e. $T_{|[L \subseteq V]} = L$. The precise statement is $T_{|[L \subseteq V]} = L \times \{(L \subseteq V)\}$.

Recall that we have done all of these to add a zero element to each fiber of the projection $p : V \setminus 0 \to \mathbb{P}(V)$. Let’s see if we have accomplished that. Since $0 \in L$ for every line $L$ in $V$, we see that the fiber $T_{|[L \subseteq V]} = L$ contains the zero element $(0, [L])$. The totality of the zero elements in the fibers will be denoted by $Z$, i.e. $Z := 0 \times \mathbb{P}(V)$. When we prove that $T$ is a line bundle on $\mathbb{P}(V)$, it will be clear that the subvariety $Z \subseteq T$ is the zero section.

Let’s begin on the left hand side now: $T \xrightarrow{β} V$. Away from the origin, i.e. when $v \neq 0$, then $β^{-1}(v)$ is a single point, $\{(v, [v])\}$. Thus, away from the origin, the varieties $T$ and $V$ are isomorphic. More precisely, $β^{-1}(V \setminus 0) \to V \setminus 0$ is an isomorphism and the following diagram commutes:

\[
\begin{array}{ccc}
\beta^{-1}(V \setminus 0) & \xrightarrow{\pi} & \mathbb{P}(V) \\
\downarrow & & \\
V \setminus 0 & \xrightarrow{p} & \mathbb{P}(V)
\end{array}
\]

Let $0 \neq v \in V$ and look at the fiber $β^{-1}(v)$ which consists of pairs $(v, [L])$ where $v \in L$. Since $v$ is nonzero, it spans the line $L$ proving that $β^{-1}(v) = \{(v, [v])\}$.

So far, we have seen that $β : T \to V$ is an isomorphism away from $0$. What about the fiber above the origin? The fiber above the origin consists of all pairs $(0, [L])$ where $0 \in L$. But this is a vacuous condition, since every linear subspace contains the origin by definition. Therefore, we see that $β^{-1}(0) = 0 \times \mathbb{P}(V)$. That is, the fiber above origin is the same as the zero divisor of the projection $π : T \to \mathbb{P}(V)$. Recall that the map $β : T \to V$ is called the blow up of $V$ at the origin. The inverse image of the blow up locus is called the exceptional divisor and denoted by $E$, in general. In this case, we rather denote it by $Z$, standing for the zero section.

The following is an important property of the universal line bundle, $O_{\mathbb{P}(V)}(-1)$.

**Proposition 3.1.** The line bundle $O_{\mathbb{P}(V)}(-1)$ has no nontrivial global sections over the projective space $\mathbb{P}(V)$. In other words, $H^0(\mathbb{P}(V), O_{\mathbb{P}(V)}(-1)) = \{0\}$.

In the proof of the above proposition, we will use of the following fact on affine varieties:

**Fact.** Suppose $P$ is a projective subvariety of the affine variety $A$. Then $P$ is a finite set of points in $A$. Furthermore, if $P$ is irreducible, then it is a point in the affine variety $A$.

**Proof of the Proposition 3.1.** Suppose $s$ is a global section of the bundle $O_{\mathbb{P}(V)}(-1)$, i.e. $s : \mathbb{P}(V) \to T$ with $π \circ s = \text{Id}_{\mathbb{P}(V)}$. Let $D$ denote the image of $s$ in $T$. Then $s : \mathbb{P}(V) \to D$ is an isomorphism. Hence $D$ is an irreducible projective variety, and so is $β(D)$. By the above fact on affine varieties, it follows that the irreducible projective variety $β(D)$ in $V$ is a single point. Let $z = β(D)$, we see that $D$ lies in the fiber $β^{-1}(z)$. However, the only fiber of dimension $≥ 1$ is $Z = β^{-1}(0)$. Thus $z = 0$ and $D \subseteq Z$. Now it follows that $D = Z$, since both $D$ and $Z$ are global sections of the bundle $T \xrightarrow{π} \mathbb{P}(V)$ and $D \subseteq Z$. This proves the section $s$ is the zero section. □
4. The Hyperplane Bundle: $O_{\mathbb{P}(V)}(1)$

In this part, we will go back to the quotient construction of the Sect. 1. Let fix an exact sequence $0 \to K \to \mathbb{U} \xrightarrow{\text{can}} V \to 0$ where $\dim K \geq 1$. Denote the map $\mathbb{P}(\text{can})$ by $\pi_{\mathbb{P}(K)}$. The maps $\text{can}$ and $\pi_{\mathbb{P}(K)}$ will be referred to as the canonical projection and the projection with center $\mathbb{P}(K)$.

We want to describe the fibers of projection $\pi_{\mathbb{P}(K)}$ first. Recall that $\mathbb{P}(K)$ sends $[L \subset \mathbb{U}]$ to $[L + K/K \subset V]$. Let $N$ be a line in $V$. The fiber above $[N \subset V]$ consists of those lines $L$ in $V$ so that $L + K/K = N$. Denote the inverse image of $N$ under the canonical projection $\mathbb{U} \xrightarrow{\text{can}} V$ by $M$. Now we have the sequence: $0 \to K \to M \xrightarrow{\text{can}|_K} N \to 0$. A point $[L \subset V]$ is in the fiber $\pi_{\mathbb{P}(K)}^{-1}([N \subset V])$ if and only if $K + L = M$. The points $[L \subset \mathbb{U}]$ with the second property form the set $\mathbb{P}(M) \setminus \mathbb{P}(K)$. Since $\dim N = 1$, the subspace $K$ is of codimension $1$ in $M$. By choosing a splitting of the sequence $0 \to K \to M \xrightarrow{\text{can}|_M} N \to 0$, one can identify the difference $\mathbb{P}(M) \setminus \mathbb{P}(K)$ by $K$, and hence $\pi_{\mathbb{P}(K)}^{-1}([N \subset V]) = \mathbb{P}(M) \setminus \mathbb{P}(K) \simeq \mathbb{A}(K)$. This identification (more properly, the local version of it) enables us to introduce an affine-line bundle structure on the projection $\mathbb{P}(U) \setminus 0 \to \mathbb{P}(V)$. Later, we will see why this is only an affine-line bundle structure and how to arrive at a line bundle structure on the projection $\mathbb{P}(U) \setminus 0 \to \mathbb{P}(V)$.

Exercise 4.1. Show that the projection $\pi_{\mathbb{P}(K)} : \mathbb{P}(U) \setminus \mathbb{P}(K) \to \mathbb{P}(V)$ can be given the structure of an affine-space bundle over the projective space $\mathbb{P}(V)$ whose fibers can be naturally identified by $\mathbb{A}(K)$.

From now on, we will fix the dimension of $K$ to be $1$. The point $[K \subset \mathbb{U}]$ will be called the origin, denoted by $0$. The projection $\pi_{\mathbb{P}(K)}$ will be denoted by $\pi_0$. Our reason to call the point $\mathbb{P}(K)$ origin will become clearer as we proceed.

Notice that the affine-line bundle $\pi_0 : \mathbb{P}(U) \setminus 0 \to \mathbb{P}(V)$ has many global sections.

Proposition 4.2. There is a bijective correspondence between the sections of the canonical projection $\mathbb{U} \xrightarrow{\text{can}} V$ and the sections of the bundle $\mathbb{P}(U) \setminus 0 \xrightarrow{\mathbb{P}(\text{can})} \mathbb{P}(V)$ via $s \mapsto \sigma := \mathbb{P}(s)$.

Proof. First, we want to show that every section $s : V \to \mathbb{U}$ induces a section $\sigma : \mathbb{P}(V) \to \mathbb{P}(U) \setminus 0$. This is easy to see.

Recall that $\pi_0 = \mathbb{P}(\text{can})$ and $\sigma := \mathbb{P}(s)$. From $\text{can} \circ s = \text{Id}_V$, it follows that $\text{Id}_{\mathbb{P}(V)} = \mathbb{P}(\text{can}) \circ \mathbb{P}(s) = \pi_0 \circ \sigma$. In addition, one observes that $0 \notin \sigma(\mathbb{P}(V))$. So we see that every section $s$ of the canonical projection can induces a section $\sigma$ of the projection $\pi_0$.

We want to prove the converse. Given a section $\sigma$ of the projection $\pi_0$, we want to build a section of the canonical projection $\mathbb{U} \xrightarrow{\text{can}} V$.

Let $D$ be the image of the section $\sigma$ in $\mathbb{P}(U)$. By definition, $0 \notin D$ and $D$ is a hypersurface in $\mathbb{P}(U)$. We know that the fibers of $\pi_0$ are the affine lines passing through the origin. Since $D$ is a section of the projection $\mathbb{P}(U) \setminus 0 \to \mathbb{P}(V)$, it meets every fiber once. Hence the degree of the hypersurface $D$ is $1$. By the Exercise 1.8, it follows that $D$ is a hyperplane in $\mathbb{P}(U)$ and there exists a hyperplane $W$ in $U$ so that $D = \mathbb{P}(W)$. By definition, $0 \notin D$, namely $K \not\subset W$. Therefore $\text{can}|_W : W \to V$ is an isomorphism of vector spaces. Now denote the inverse of $\text{can}|_W$ by $s$. By definition, $s : V \to W$ is a section of the canonical projection $\text{can} : \mathbb{U} \to V$.

Notice that the bundle $\pi_0 : \mathbb{P}(U) \setminus 0 \to \mathbb{P}(V)$ has no preferred section. However, every vector bundle comes with a distinguished section, namely the zero section. This reminds us that bundle we have got so far is an affine-line bundle without a linear structure. In order to put the structure of a line bundle on $\pi_0 : \mathbb{P}(U) \setminus 0 \to \mathbb{P}(V)$, we have to choose a section $s_0$ and declare it as the zero section. By the above discussion, this is equivalent to choosing a section $s_0$ of the canonical projection $\mathbb{U} \xrightarrow{\text{can}} V$, hence a splitting of the vector space $U$. We can do so. Or, we start from the
beginning with \( U := V \oplus k \). Clearly, our previous discussions remain valid. Furthermore, we have a distinguished section \( s_0 : V \to V \oplus k \) defined by \( v \in V \mapsto (v, 0) \). Denote the projections onto each factor by \( \text{pr}_i \) for \( i = 1, 2 \). We see that the canonical projection is \( \text{pr}_1 \), the vector space \( U \) splits via the map \( s_0 \) and under the effect of the zero section \( \sigma_0 \), the projective space \( \mathbb{P}(V) \) becomes a linear subspace of the vector space \( \mathbb{P}(V \oplus k) \).

Now we want to translate the Proposition 4.2 to our new setup:

**Proposition 4.3.** \( H^0(\mathbb{P}(V), O_{\mathbb{P}(V)}(1)) = V^\vee \).

**Proof.** In Proposition 4.2, we constructed a bijective correspondence between the sections of the bundle \( \mathbb{P}(V \oplus k) \setminus 0 \to \mathbb{P}(V) \) and the projection \( \text{pr}_1 : V \oplus k \to V \). We saw that \( \sigma : \mathbb{P}(V) \to \mathbb{P}(V \oplus k) \setminus 0 \) is a section of the bundle if and only if there exists a section \( s \) of the canonical projection \( V \oplus k \to V \) so that \( \sigma = \mathbb{P}(s) \).

Now we want to build on this result. Given a linear function \( f : V \to k \), define the section \( s \) and its projectivization \( \sigma \) by setting \( s := \text{Id} \oplus f (v) \in V \to (v, f(v)) \) and \( \sigma = \mathbb{P}(s) \). Obviously, \( \sigma \) is a section of the bundle \( \mathbb{P}(U) \setminus 0 \to \mathbb{P}(V) \). In other words, \( \sigma \) is a global section of the line bundle \( O_{\mathbb{P}(V)}(1) \).

Conversely, pick a global section \( \sigma \) of \( O_{\mathbb{P}(V)}(1) \). By the above statements, there is a section \( s \) of the canonical projection \( U \to V \). Set \( f := \text{pr}_2 \circ s \). Notice that \( \text{pr}_1 \circ s = \text{can} \circ s = \text{Id}_V \). Therefore we see that \( s(v) = (v, f(v)) \) for \( v \in V \). By its definition, the linear map \( f : V \to k \) is a linear function on \( V \), i.e. \( f \in V^\vee \). \( \square \)

The name hyperplane bundle comes from the fact that for a nonzero section \( \sigma \), the locus where the section \( \sigma \) meets the zero section of the line bundle \( \mathbb{P}(V \oplus k) \setminus 0 \to V \) is a hyperplane in \( \mathbb{P}(V) \). Denote the image of the section \( \sigma \) by \( \mathbb{P}(W) \). Since the section \( \sigma \) is nontrivial, \( \mathbb{P}(W) \neq \mathbb{P}(V) \). Thus the intersection of \( \mathbb{P}(W) \) and \( \mathbb{P}(V) \) equals \( \mathbb{P}(W \cap V) \) and is a hyperplane in \( \mathbb{P}(V) \).

**Exercise 4.4.** Conclude the same statement using the section \( s \) and the linear function \( f \) rather than the section \( \sigma \). Indeed, show that \( \mathbb{P}(W \cap V) = \mathbb{P}(\ker f) \).

### 5. The Others: \( O_{\mathbb{P}(V)}(+/- d) \) for \( d > 0 \)

In this section, we will discuss how to obtain the line bundles \( O_{\mathbb{P}(V)}(-/ + d) \) for \( d > 0 \) using the universal line bundle, \( O_{\mathbb{P}(V)}(-1) \), and the hyperplane bundle, \( O_{\mathbb{P}(V)}(1) \), together with the \( d \)-th Veronese embedding \( \mathbb{P}(V) \to \mathbb{P}(\text{Sym}^d V) \). The diagram

\[
O_{\mathbb{P}(V)}(d) \to O_{\mathbb{P}(\text{Sym}^d V)}(1) \\
\mathbb{P}(V) \to \mathbb{P}(\text{Sym}^d V)
\]

is a brief summary of what we will see in this section.

In other words, the objective of this section is to work out the above diagram *inside-out.*

Recall that the map \( V \to \text{Sym}^d V \) sends a vector \( v \in V \) to its \( d \)-th tensor power \( v^{\otimes d} \in \text{Sym}^d V \subset V^{\otimes d} \). The projective counterpart of this map is the \( d \)-th Veronese embedding and sends the line \( L \) in \( V \) to the line \( L^{\otimes d} \) in the \( d \)-th symmetric product \( \text{Sym}^d V \), i.e. \( [L] \in \mathbb{P}(V) \mapsto [L^{\otimes d}] \in \mathbb{P}(\text{Sym}^d V) \subset \mathbb{P}(V^{\otimes d}) \).

Any nontrivial section \( s \) of the hyperplane bundle, \( O_{\mathbb{P}(\text{Sym}^d V)}(1) \), cuts the hyperplane, \( \text{div}(s) \), in the projective space \( \mathbb{P}(\text{Sym}^d V) \) as explained in Sect.0.2. Therefore, any such section induces a linear relation between the \( d \)-th tensor (equivalently, symmetric) powers, or equivalently, a degree \( d \) homogeneous relation in \( \mathbb{P}(V) \).
In other words, any section of the bundle, \( \theta_d^*(\mathcal{O}_{\mathbb{P}(\text{Sym}^dV)}(1)) \) cuts a degree \( d \) hypersurface in the projective space \( \mathbb{P}(V) \). The pullback is the same as the line bundle line bundle, \( \mathcal{O}_{\mathbb{P}(V)}(d) \). The fibers of the bundle, \( \mathcal{O}_{\mathbb{P}(V)}(d) \), parameterize the degree \( d \) forms on the lines of \( V \). As a consistency check, we verify that the pullback is the \( d \)th multiple of the hyperplane bundle on \( \mathbb{P}(V) \), i.e. \( \theta_d^*(\mathcal{O}_{\mathbb{P}(\text{Sym}^dV)}(1)) = \mathcal{O}_{\mathbb{P}(V)}(d) = \mathcal{O}_{\mathbb{P}(V)}(1)^\otimes d \). By definition, \( \mathcal{O}_{\mathbb{P}(V)}(1)[L] = L^\vee \) and \( \mathcal{O}_{\mathbb{P}(V)}(d) = \text{Sym}^dL^\vee = (L^\vee)^\otimes d \). Similarly, \( \mathcal{O}_{\mathbb{P}(\text{Sym}^dV)}(1)[W] = W^\vee \) for an arbitrary line \( W \) in \( \text{Sym}^dV \) and \( \mathcal{O}_{\mathbb{P}(\text{Sym}^dV)}(1)[L^\otimes d] = (L^\otimes d)^\vee \) for a line \( W = L^\otimes d \) in the image of \( V \to \text{Sym}^dV \). We see that \( \mathcal{O}_{\mathbb{P}(V)}(d)[L] = (L^\vee)^\otimes d = (L^\otimes d)^\vee = \mathcal{O}_{\mathbb{P}(\text{Sym}^dV)}(1)[L^\otimes d] \) proving

\[
\theta_d^*(\mathcal{O}_{\mathbb{P}(\text{Sym}^dV)}(1)) = \mathcal{O}_{\mathbb{P}(V)}(d).
\]

Dualizing, we get the other half \( -d < 0 \):

\[
\theta_d^*(\mathcal{O}_{\mathbb{P}(\text{Sym}^dV)}(-1)) = \mathcal{O}_{\mathbb{P}(V)}(-d).
\]

A fancier way to say what we have done above is as follows: Take \( d \)th symmetric power of the inclusion \( \mathcal{O}_{\mathbb{P}(V)}(-1) \to V^\otimes \mathcal{O} \) and notice that the restriction of the universal sequence \( \mathcal{O}_{\mathbb{P}(\text{Sym}^dV)}(-1) \to \text{Sym}^dV \otimes \mathcal{O} \) to the image of the Veronese embedding is the same as the previous sequence. Similarly, one can conclude arrive the corresponding statement about \( \mathcal{O}_{\mathbb{P}(V)}(d) \) using the dual sequences \( V^\vee \otimes \mathcal{O} \to \mathcal{O}_{\mathbb{P}(V)}(1) \) (take \( d \)th symmetric power) and \( \text{Sym}^dV \to \text{OSP}(1) \) (restrict to the image of the Veronese embedding). The resulting sequence of both these two operations is the sequence \( \text{Sym}^dV \otimes \mathcal{O} \to \mathcal{O}_{\mathbb{P}(V)}(d) \) on the projective space \( \mathbb{P}(V) \). One can interpret this sequence as restricting the \( d \) forms on the vector space \( V \) to those on the lines of \( V \). Similar to Proposition 4.3, we have the generalization whose proof is left as an exercise:

**Proposition 5.1.** \( H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(d)) = \text{Sym}^dV^\vee \).

Obviously, the there is an injective linear map from the \( d \)th symmetric dual \( \text{Sym}^dV^\vee \) to the global sections of the line bundle \( \mathcal{O}_{\mathbb{P}(V)}(d) \). As before, the content of the theorem is that this map is surjective.

For the negative values, \( -d < 0 \), we see that the proof of the Proposition 3.1 goes through, due to the fact that the total space of the line bundle, \( \mathcal{O}_{\mathbb{P}(V)}(-d) \), has a map to an affine variety, namely \( \text{Sym}^dV \). This proves

**Proposition 5.2.** \( H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(-d)) = \{0\} \).

We want to summarize and comment on the results of this section briefly.

Hyperplane sections of the image of the Veronese embedding \( \mathbb{P}(V) \hookrightarrow \mathbb{P}(\text{Sym}^dV) \) are degree \( d \) hypersurfaces in \( \mathbb{P}(V) \). Furthermore, any degree \( d \) hypersurface in \( \mathbb{P}(V) \) is given as a hyperplane section of this embedding. In other words, degree \( d \) hypersurfaces are sent to hyperplane sections in \( \mathbb{P}(\text{Sym}^dV) \).

This is a common technique to reduce certain questions on hypersurfaces of degree \( d \) in \( \mathbb{P}(V) \) to that of hyperplane sections of the image of Veronese embedding.

The bundle theoretic way to say the above is the equality

\[
\theta_d^*(\mathcal{O}_{\mathbb{P}(\text{Sym}^dV)}(1)) = \mathcal{O}_{\mathbb{P}(V)}(d) = \mathcal{O}_{\mathbb{P}(V)}(1)^\otimes d.
\]

The space of global sections of the line bundles, \( \mathcal{O}_{\mathbb{P}(V)}(-d) \) and \( \mathcal{O}_{\mathbb{P}(V)}(d) \) are seen to be trivial, \( \{0\} \) and the \( d \)th symmetric dual, \( \text{Sym}^dV^\vee \), respectively.

This finishes our discussion on line bundles of projective spaces using projective geometry.
6. LOCAL COORDINATES

So far, we have not checked that the bundles described in Sect. 3.1 and Sect. 4 are line bundles. This section is devoted to the proof of this result.

Lemma 6.1.  
(1) The projection $p : V \setminus \emptyset \to \mathbb{P}(V)$ is a principal $\mathbb{G}_m$-bundle.

(2) The projections $\pi : T \to \mathbb{P}(V)$ and $\pi_0 : \mathbb{P}(V \oplus k) \to \mathbb{P}(V)$ have the structure of a line bundle.

Proof. First of all, (2) implies (1). The fancy way to say that is the equivariant decomposition of the affine line $\mathbb{A}^1$ with respect to $\mathbb{G}_m$ is $\emptyset \cup \mathbb{G}_m$. Therefore the total space of a line bundle has analogous equivariant decomposition with respect to the multiplicative group $\mathbb{G}_m$, first piece being the zero section of the line bundle and the second being the the total space of the principal $\mathbb{G}_m$-bundle defining the line bundle. In the case of the universal line bundle, the principal $\mathbb{G}_m$-bundle of consideration is the one defined by (1), $\mathcal{E}_{\mathbb{G}_m}$. For the hyperplane bundle, this is also true, but needs a bit of thinking.

The other way to see how (2) implies (1) is require the extra assumption that the local sections of consideration in (2) never vanish during the proof of the statement (2). This is identical what is said in the previous paragraph.

Let’s prove the first assertion in (2) first. Assume $\dim V = n + 1$, choose a basis $e_0, e_1, \ldots, e_n$ of $V$ and a dual basis of $V^\vee$. Introduce the coordinates $(\nu_0, \nu_1, \ldots, \nu_n)$ to denote the point $\nu_0 e_0 + \nu_1 e_1 + \cdots + \nu_n e_n \in V$, the homogeneous coordinates $[x_0 : x_1 : \cdots : x_n]$ on $\mathbb{P}(V)$ to denote the line $L$ spanned by the vector $x = x_0 e_0 + x_1 e_1 + \cdots + x_n e_n \in V$. Let $U_i \subset \mathbb{P}(V)$ be the open set for which $x_i \neq 0$. On the open set $U_i$, we will use the coordinates $\frac{x_0}{x_i}, \frac{x_1}{x_i}, \ldots, \frac{x_n}{x_i}$ where $\frac{x_i}{x_i} = 1$ is omitted. Notice that this choice of coordinates identifies $U_i$ with the affine $n$-space.

The blowup $T$ of $V$ at the origin is a determinantal variety, meaning that it is defined by vanishing of some determinants. Precisely speaking,

\[ v \in L \iff v \text{ and } x \text{ are linearly dependent} \]

\[ \iff \text{rank } \begin{bmatrix} v \\ x \end{bmatrix} \leq 1 \]

\[ \iff \text{For all } i, j = 0, \ldots, n, \; v_i x_j = v_j x_i. \]

After describing the total space $T$ so concretely, we could continue and write down the local trivializations $h_i$ on the open set $U_i$ for $i = 0, \ldots, n$ and the transition functions $g_{ij}$ on the overlaps $U_i \cap U_j$ for $i \neq j = 0, \ldots, n$.

On the open set $U_i$, the homogeneous variable $x_i$ is nonzero by definition. Coupling this with the last of the above assertions, we see that, on $U_i$, one has $v_i = \frac{x_i}{x_i} v_i$. In other words, the value of $v_j$, $j \neq i$ is determined by the coordinates $\frac{x_i}{x_i}$ and $v_i$. This motivates us to define the local trivialization $h_i : \pi^{-1}(U_i) \to k \times U_i$ as $h_i([v, [x]] := ([v_i, [x]])$. The first part of the local trivialization $h_i$ on $U_i$ gives a local section $s_i$ on $U_i$ defined by $s_i([v, [x]]) = v_i$. On the overlap $U_i \cap U_j$, the transition function $g_{ij}$ should satisfy the equation $s_i = g_{ij} s_j$ for $i \neq j = 0, \ldots, n$. Reverting the equation $v_j = \frac{x_i}{x_j} v_i$, we see that $v_i = \frac{x_i}{x_j} v_j$, which on the level of local sections says that $s_i = \frac{x_i}{x_j} s_j$, and thus $g_{ij} = \frac{x_i}{x_j}$. To prove the second assertion of (2), call the extra coordinate as $x_{n+1}$. Then set the coordinates on the open set $U_i$, $i = 0, \ldots, n$ as before, then on the inverse image $\pi_0^{-1}(U_i) = \{ x_i \neq 0 \} \mathbb{P}(V \oplus k)$, the coordinates will be $\frac{x_0}{x_i}, \frac{x_1}{x_i}, \ldots, \frac{x_n}{x_i}$ coming from $U_i$ and $\frac{x_{n+1}}{x_i}$ from the fiber. Hence, we define the local trivialization $h_i : \pi_0^{-1}(U_i) \to k \times U_i$ by $h_i([x_0, x_1, \ldots, x_n, x_{n+1}]) := (\frac{x_{n+1}}{x_i}, [x_0, x_1, \ldots, x_n]).$ For $i = 0, \ldots, n$, the local section $s_i$ on $U_i$ is defined accordingly by $s_i([x_0, x_1, \ldots, x_n, x_{n+1}]) := \frac{x_{n+1}}{x_i}$. For $i \neq j = 0, \ldots, n$, the transition function $g_{ij}$ is seen to be $\frac{x_i}{x_j}$. 

We can now conclude that \( \pi : T \to \mathbb{P}(V) \) and \( \pi_0 : \mathbb{P}(V \oplus k) \setminus 0 \to \mathbb{P}(V) \) define line bundles, and looking at their transition functions, \( \frac{a_i}{x_i} \) and \( \frac{a_j}{x_i} \) for \( i \neq j = 0, \ldots, n \), we conclude that these two line bundles are duals of each other.

As a consequence of this result, for arbitrary integer \( d \), we see that the acclaimed bundle \( \mathcal{O}_{\mathbb{P}(V)}(d) \) is a line bundle with transition functions, \( g_{ij} = \frac{x_i}{x_j}^d \).

**Exercise 6.2.** Construct the duality between the universal line bundle and the hyperplane bundle in the spirit of the paper using their total spaces without using the transition functions. (Hint: Cross-ratio.)

### 7. Multiplicative Group Quotients

This section aims to show how one can build all the line bundles on the projective space \( \mathbb{P}(V) \) using the principal \( \mathbb{G}_m \)-bundle \( E_{\mathbb{G}_m} \). Recall that, \( E_{\mathbb{G}_m} \) is defined by the projection \( p : V \setminus 0 \to \mathbb{P}(V) \) with transition functions \( g_{ij} = \frac{x_i}{x_j} \). Given a character \( \chi : \mathbb{G}_m \to \mathbb{G}_m \), let \( L_\chi \) be the 1-dimensional vector space on which \( \mathbb{G}_m \) acts on the left by \( \lambda \cdot v := \chi(\lambda)v \). Every character \( \chi : \mathbb{G}_m \to \mathbb{G}_m \) is of the form \( \lambda \mapsto \lambda^d \) for some integer \( d \). The number \( d \) is called the the degree of the character \( \chi \) and denoted by \( \deg \chi \). Now, consider the associated line bundle obtained from the principal \( \mathbb{G}_m \)-bundle by its representation \( L_\chi \). Resulting line bundle will be denoted by \( \mathcal{O}_{\mathbb{P}(V)}(\chi) \). (This is not a standard notation, however it works for us.)

The highlight of the section is the equality, \( \mathcal{O}_{\mathbb{P}(V)}(\chi) = \mathcal{O}_{\mathbb{P}(V)}(-\deg \chi) \). That is, the line bundle associated to the character, \( \chi \), is the line bundle whose degree is equal to minus the degree of the character.

Following [R], we will introduce the line bundle \( \mathcal{O}_{\mathbb{P}(V)}(\chi) \) as a quotient of the multiplicative group \( \mathbb{G}_m \). The second chapter of Reid’s book [R] is one of my favorite readings on vector bundles. The chapter is entitled ‘Rational Scrolls’ and studies projective bundles on the projective line.

We want to point out the sign difference between Reid’s definitions and ours. Given base variety \( B \) and a vector bundle \( V \) on \( B \), denote the variety of 1-dimensional linear subspaces of \( V \) by \( \mathbb{P}_B(V) \). Then the rational scroll considered by Reid, \( \mathbb{P}(a_1, a_2, \ldots, a_n) \) is given by the projectivization of the vector bundle \( V = \mathcal{O}_{\mathbb{P}^1}(-a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(-a_2) \oplus \cdots \mathcal{O}_{\mathbb{P}^1}(-a_n) \) on the base \( \mathbb{P}^1 \), i.e. \( \mathbb{P}(a_1, a_2, \ldots, a_n) = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(-a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(-a_2) \oplus \cdots \mathcal{O}_{\mathbb{P}^1}(-a_n)) \).

Sticking to the classical conventions, we assume that the multiplicative group \( \mathbb{G}_m \) acts on the vector space \( V \) on the right and on the vector space \( k \) on the left. Fix an integer \( d \) and hence a character \( \chi : \lambda \mapsto \lambda^d \). On the space \( \mathbb{P}(V) \setminus 0 \times k \), \( \lambda \in \mathbb{G}_m \) acts via:

\[
(v, t) \cdot \lambda := (v\lambda, \chi(\lambda)^{-1}t) = (\lambda v, \lambda^{-d}t)
\]

Of course, the map \( \mathbb{P}(V) \setminus 0 \times k \to \mathbb{P}(V) \setminus 0 \) preserves the \( \mathbb{G}_m \)-action both sides, and hence the map descends to the quotients. The quotient on the right is obviously the projective space \( \mathbb{P}(V) \). Our claim is that the one on the left is the total space of the line bundle \( \mathcal{O}_{\mathbb{P}(V)}(-d) \).

For this, we will calculate the transition functions given by the line bundles above.

As before, let \( U_i \) denote the open set on which the homogeneous coordinate \( x_i \) is nonzero, let \( h_i \) denote the local trivialization on the coordinate patch \( U_i \) and \( s_i \) denote the section obtained from the trivialization \( h_i \) on \( U_i \). The transition function \( g_{ij} \) describes how the fiber coordinate changes passing from the patch \( U_i \) to the patch \( U_i \). Let’s assume \( [x] \in U_i \), thus can be written as \( [x_0 : \ldots : x_n] \), since on the coordinate patch, the coordinates are \( x_0 : \ldots : x_n \), we let \( \lambda = \frac{x_i}{x_0} \), so \( s_i = (\frac{x_i}{x_0})^{-d} s_j \). Hence the transition function \( g_{ij} \) is \( (\frac{x_i}{x_0})^{-d} \), proving that the line bundle \( \mathcal{O}_{\mathbb{P}(V)}(\chi) \) is the line bundle on degree \( -\deg \chi \), \( \mathcal{O}_{\mathbb{P}(V)}(-\deg \chi) \).
Exercise 7.1. Construct the global sections of the line bundle $\mathcal{O}_{\mathbb{P}(V)}(d), d \in \mathbb{Z}$ using the description given in this section. Conclude that line bundles of negative degree, $d < 0$, have no nontrivial global sections.

8. CONCLUSION

Projective spaces are very rich in terms of structure. As a result of this, the object related to projective spaces have many different descriptions. To cite only few, recall that projective spaces are toric varieties and at the same time, homogeneous manifolds.

Toric varieties have been on the forefront of mathematics in the last few decades with growing number of applications. Projective spaces are one of the main examples of toric varieties. On a toric variety, it is possible to determine all the line bundles looking at the combinatorial data defining the toric variety. Fulton’s book [F] gives a nice introduction to toric varieties and covers the material on line bundles as well.

Another aspect of projective space is that it is a homogeneous manifold. It is one of the simplest examples of the so-called partial flag manifolds. It can be identified with a quotient of the general linear group by a particular type of subgroup. This direction leads us to semisimple lie groups, root systems, parabolic subgroups, and other related group-theoretic constructs. The climax of this theory in the direction of our intentions is the celebrated Borel–Weil–Bott Theorem describing the homogeneous vector bundles and their cohomology groups via group theory or vica versa. The book Penrose Transform: ... [BE] is a very lively introduction to the concepts of the field and the proof of the Borel–Weil–Bott Theorem.

A transverse direction to our presentation is the language one works in. There are at least two: Differential Geometry and Algebraic Geometry.

Although our fundamental presentation of line bundles has a somewhat differential geometric flavor since we represented line bundles by describing their total spaces, our exposition is not fully complete because we have not calculated any related differential geometric object, such as connections etc. One obvious exercise is to calculate the connections and their curvatures on each of the line bundles, $\mathcal{O}_{\mathbb{P}(V)}(d), d \in \mathbb{Z}$.

I have planned to look at the objects in this paper as coherent sheaves, so far my plan has not been realized. This leaves the reader the opportunity to cover the absence by her/his own effort providing a better understanding of the subject via algebraic geometry. One can also compare our presentation with that of Hartshorne’s [Ha].

The wisdom of learning mathematics is very well expressed in the words “There is no royal path to mathematics”. I suggest the interested reader should go over the material contained in the paper, transcribing the coordinate-free arguments by using local coordinates, the calculations involving the local computations with direct use of the knowledge at hand without reference to coordinates. Another direction one could take is to try to work out the results given in the paper for Grassmann varieties other than projective spaces.

I hope this paper would help the reader develop her/his knowledge of the subject matter, and her/his views on whole mathematics as well.

REFERENCES


